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Elliptic solutions of the restricted Toda chain, Lamé polynomials and generalization of the elliptic Stieltjes polynomials

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Abstract
We construct new families of elliptic solutions of the restricted Toda chain. The main tool is a special (so-called Stieltjes) ansatz for the moments of corresponding orthogonal polynomials. We show that the moments thus obtained are related to three types of Lamé polynomials. The corresponding orthogonal polynomials can be considered as a generalization of the Stieltjes–Carlitz elliptic polynomials.

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1. Introduction

The Toda chain [21] is an example of a completely integrable classical many-particle system with highly regular behavior. Initially this model was proposed by Toda from physical considerations (as an example of a many-particle system without quasi-stochastization). However, it was soon recognized that this model has many applications in different branches of physics and mathematics.

In mathematical physics, the Toda chain is usually associated with tri-diagonal (Jacobi) matrices and corresponding orthogonal polynomials $P_n(x; t)$ depending on an additional (time) parameter [3, 18, 21].

In particular, the Toda chain provides a possibility to construct explicit families of orthogonal polynomials and to study their properties.

In [17], a method was proposed to construct explicit solutions of the restricted Toda chain starting from a special polynomial ansatz for the moments of orthogonal polynomials.

In this paper, we generalize this ansatz and obtain new classes of explicit solutions of the Toda chain which are related on the one hand with the Lamé polynomials and on the other hand with the Stieltjes–Carlitz elliptic orthogonal polynomials.

We recall some basic definitions and results concerning the relations between Toda chain and orthogonal polynomials [3, 17].

We consider the Toda chain equations in the form [21]

$$\dot{u}_n = u_n(b_n - b_{n-1}), \qquad \dot{b}_n = u_{n+1} - u_n \tag{1.1}$$

with restriction

$$u_0 = 0, \tag{1.2}$$

where the dot indicates differentiation with respect to t. In what follows we will call equations (1.1) with restriction (1.2) *the restricted Toda chain* (TC) equations.

Let $P_n(x;t)$ be orthogonal polynomials satisfying the three-term recurrence relation

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x) \tag{1.3}$$

with initial conditions

$$P_0 = 1, \qquad P_1(x) = x - b_0. \tag{1.4}$$

We will assume that $u_n \neq 0, n = 1, 2, \ldots$. By the Favard theorem [5], there exists a nondegenerate linear functional σ such that the polynomials $P_n(x)$ are orthogonal with respect to it:

$$\sigma(P_n(x)P_m(x)) = h_n\delta_{nm}, \tag{1.5}$$

where h_n are normalization constants. The linear functional σ can be defined through its moments

$$c_n = \sigma(x^n), \qquad n = 0, 1, \ldots. \tag{1.6}$$

It is usually assumed that $c_0 = 1$ (standard normalization condition), but we will not assume this condition in the following, i.e. it is supposed that c_0 is an arbitrary nonzero parameter.

Introduce the Hankel determinants

$$D_n = \det(c_{i+j})_{i,j=0,\ldots,n-1}, \qquad D_0 = 1, \qquad D_1 = c_0. \tag{1.7}$$

Then the polynomials $P_n(x)$ can be uniquely represented as [5]

$$P_n(x) = \frac{1}{D_n}\begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}. \tag{1.8}$$

The normalization constants are expressed as

$$h_n = \frac{D_{n+1}}{D_n}, \qquad h_0 = D_1 = c_0. \tag{1.9}$$

While the recurrence coefficients u_n satisfy the relation

$$u_n = \frac{h_n}{h_{n-1}} = \frac{D_{n-1}D_{n+1}}{D_n^2}. \tag{1.10}$$

Thus we have

$$h_n = c_0 u_1 u_2 \cdots u_n. \tag{1.11}$$

Assume now that the polynomials $P_n(x;t)$ depend on a real parameter t through their recurrence coefficients $u_n(t), b_n(t)$. Then the restricted Toda chain equations (RTE) are equivalent to the condition

$$\dot{P}_n(x;t) = -u_n P_{n-1}(x;t). \tag{1.12}$$

It is possible to choose the initial moment $c_0(t)$ (normalization) such that the RTE are equivalent to the condition

$$\dot{c}_n = c_{n+1}, \tag{1.13}$$

i.e.

$$c_n(t) = \frac{\mathrm{d}^n c_0(t)}{\mathrm{d}t^n}. \tag{1.14}$$

In this case the Hankel determinants $D_n = D_n(t)$ have the form

$$D_n = \det\left(c_0^{(i+k)}\right)_{i,k=0,\dots,n-1}, \qquad D_0 = 1, \qquad D_1 = c_0, \tag{1.15}$$

where $c_0^{(j)}$ means the jth derivative of $c_0(t)$ with respect to t.

Under this condition, the RTE are equivalent also to the equations

$$\frac{\mathrm{d}^2 \log D_n}{\mathrm{d}t^2} = \frac{D_{n-1}D_{n+1}}{D_n^2}, \qquad n = 1, 2, \dots. \tag{1.16}$$

(Equations (1.16) are equivalent to the Hirota bilinear form [9] for the RTE.)

Note also that for the Hankel determinants of the form (1.15) we have two useful relations

$$b_n = \frac{\dot{D}_{n+1}}{D_{n+1}} - \frac{\dot{D}_n}{D_n} \tag{1.17}$$

and

$$\dot{h}_n = h_n b_n. \tag{1.18}$$

In particular, for $n = 0$ we have from (1.18)

$$b_0 = \frac{\dot{c}_0}{c_0}. \tag{1.19}$$

Relation (1.19) allows us to restore $c_0(t)$ if the recurrence coefficient $b_0 = b_0(t)$ is known explicitly from Toda chain solutions (1.1).

2. Toda chain and Christoffel transform

Let $b_n(t), u_n(t)$ be a solution to the restricted Toda chain (1.1) corresponding to the zero moment $c_0(t)$. Let $P_n(x;t)$ be a set of monic orthogonal polynomials satisfying conditions (1.3) and (1.12).

Consider the simplest Christoffel transform of the orthogonal polynomials $P_n(x;t)$:

$$\tilde{P}_n(x;t) = \frac{P_{n+1}(x;t) - A_n(t)P_n(x;t)}{x}, \tag{2.1}$$

where

$$A_n(t) = \frac{P_{n+1}(0;t)}{P_n(0;t)}.$$

It is well known (see, e.g. [17, 24]) that transformation (2.1) gives new orthogonal polynomials $\tilde{P}_n(x;t)$ which again satisfy Toda conditions (1.12) with recurrence coefficients

$$\tilde{u}_n = u_n\frac{A_n}{A_{n-1}}, \qquad \tilde{b}_n = b_{n+1} + A_{n+1} - A_n. \tag{2.2}$$

The new coefficients $\tilde{u}_n, \tilde{b}_n$ satisfy the same restricted Toda chain equations (1.1). The new zero moment function $\tilde{c}_0(t)$ will be $\tilde{c}_0(t) = \dot{c}_0(t) = c_1(t)$, i.e. the Christoffel transform (2.1) and (2.2) are equivalent to a simple shift $c_n(t) \to c_{n+1}(t)$ of all the moments. Such a shift is well known in theory of the so-called qd-algorithm (see, e.g. [4]) which was in fact discovered by Stieltjes in [19].

If the polynomials $P_n(x;t)$ are orthogonal on the interval $[a,b]$ (either finite or infinite) of the real line with some weight function $w(x;t)$

$$\int_a^b P_n(x;t)P_m(x;t)w(x;t)\,\mathrm{d}x = h_n(t)\delta_{nm}, \tag{2.3}$$

then the transformed polynomials $\tilde{P}_n(x;t)$ will be orthogonal on the same interval $[a,b]$ with respect to the new weight function

$$\tilde{w}(x) = xw(x). \tag{2.4}$$

In particular, assume that polynomials $P_n(x;t)$ are orthogonal on a grid x_s with concentrated masses M_s:

$$\sum_{s=-\infty}^{\infty} M_s P_n(x_s;t)P_m(x_s;t) = h_n(t), \delta_{nm}, \tag{2.5}$$

then the transformed polynomials $\tilde{P}_n(x;t)$ will be orthogonal on the same grid with concentrated masses $\tilde{M}_s = x_s M_s$:

$$\sum_{s=-\infty}^{\infty} x_s M_s \tilde{P}_n(x_s;t)\tilde{P}_m(x_s;t) = \tilde{h}_n(t)\delta_{nm}. \tag{2.6}$$

Note that the reciprocal transformation (the so-called Geronimus transform [24]) is

$$P_n(x;t) = \tilde{P}_n(x;t) - B_n(t)\tilde{P}_{n-1}(x;t), \tag{2.7}$$

where

$$B_n(t) = u_n(t)\frac{P_{n-1}(0;t)}{P_n(0;t)}.$$

In terms of the functions $A_n(t)$, $B_n(t)$, the Toda chain equations (1.1) can be presented in an equivalent form as

$$\dot{A}_n = A_n(B_n - B_{n+1}), \qquad \dot{B}_n = B_n(A_{n-1} - A_n), \qquad B_0 = 0. \tag{2.8}$$

Sometimes the form (2.8) is more convenient because the recurrence coefficients u_n, b_n can be expressed in terms of $A_n(t)$, $B_n(t)$ as

$$u_n(t) = B_n(t)A_{n-1}(t), \qquad b_n(t) = -A_n(t) - B_n(t). \tag{2.9}$$

We also need an explicit expression for the 'shifted' Hankel determinant

$$\tilde{D}_n(t) = \begin{vmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{vmatrix}.$$

From (1.8) it is seen that

$$\tilde{D}_n = (-1)^{n+1} D_n P_n(0;t). \tag{2.10}$$

Formula (2.10) may be useful for calculating the shifted determinant $\tilde{D}$ if the polynomial $P_n(0;t)$ is known explicitly (e.g. expressible in terms of hypergeometric functions). In what follows we shall use this observation.

3. The Stieltjes polynomial ansatz and elliptic moments

In what follows we will assume that the moments $c_n(t)$ are defined through the zero moment $c_0(t)$ by the condition $c_n(t) = \frac{\mathrm{d}^n c_0(t)}{\mathrm{d}t^n}$ which is equivalent to the restricted Toda chain equations (1.12) for the corresponding orthogonal polynomials $P_n(x;t)$. In [17] a separation of variables ansatz was proposed

$$c_n(t) = T_n(y(t))c_0(t), \qquad n = 0, 1, \ldots, \tag{3.1}$$

where $T_n(y(t))$ is a polynomial of exact degree n in some (unknown) variable $y(t)$ with coefficients not depending on t. This leads to a system of orthogonal polynomials of the Sheffer class: Meixner, Pollaczek, Krawtchouk, Laguerre, Hermite and Charlier. In turn, these systems of orthogonal polynomials, considered from the Toda chain point of view, give rise to combinatorial numbers such as Euler, Bell, etc.

In this paper we generalize this ansatz in the following way. Instead of (3.1) we demand that

$$c_{2n}(t) = T_n(y(t))c_0(t), \qquad n = 0, 1, \ldots, \tag{3.2}$$

where, again, $T_n(y)$ are polynomials of nth degree in y with coefficients not depending on t. In other words, we only require the polynomial property (3.2) for the *even* moments $c_{2n}(t)$; the odd moments $c_{2n+1}(t)$ in general, may not satisfy this property.

This problem was first considered by Stieltjes [19] who found several explicit examples of the functions $c_0(t)$ satisfying condition (3.2). Stieltjes also demonstrated a remarkable relation of this property with the possibility to construct explicitly the corresponding continued fractions. It is reasonable therefore to call (3.2) the Stieltjes ansatz.

Our main goal will be

(i) to find all possible solutions $c_0(t), c_1(t), \ldots, c_n(t), \ldots$ corresponding to the Stieltjes ansatz (3.2);
(ii) to find explicitly the corresponding Hankel determinants $D_n(t)$ constructed from moments $c_n(t)$ by (1.7);
(iii) to find explicitly the recurrence coefficients $u_n(t), b_n(t)$ which are solutions of the restricted Toda chain (1.1) corresponding to the function $c_0(t)$;
(iv) to find explicitly the corresponding orthogonal polynomials $P_n(x;t)$ and their orthogonality measure.

We will see that apart from the well-known orthogonal polynomials (such as Meixner, Krawtchouk, Hermite,...) there appear new types of orthogonal polynomials related to elliptic functions. These orthogonal polynomials can be considered as a natural generalization of the elliptic Stieltjes–Carlitz polynomials (for the latter see, e.g. [12, 13, 16]).

From (1.14) we get that the Stieltjes ansatz (3.2) is equivalent to

$$L^{2n}\{\phi(y)\} = T_n(y)\phi(y), \tag{3.3}$$

where $\phi(y) \equiv c_0(t)$ and L is a first-order differential operator

$$Lf(y) = s(y)f'(y) \tag{3.4}$$

with

$$s(y) \equiv \mathrm{d}y/\mathrm{d}t \tag{3.5}$$

(we consider here $\mathrm{d}y/\mathrm{d}t$ as a function of y instead of t, this is possible because the function $y(t)$ is assumed to be invertible).

In turn, condition (3.3) is equivalent to

$$L^2\{T_n(y)\phi(y)\} = T_{n+1}(y)\phi(y), \quad n = 0, 1, \ldots \tag{3.6}$$

or, in explicit form

$$T_{n+1}(y) = A(y)T_n''(y) + B(y)T_n'(y) + C(y)T_n(y), \qquad n = 0, 1, \ldots \tag{3.7}$$

where

$$\begin{aligned} &A(y) = s^2(y), \qquad B(y) = 2s^2(y)\psi(y) + s(y)s'(y), \\ &C(y) = s^2(y)(\psi'(y) + \psi^2(y)) + \psi(y)s(y)s'(y) \end{aligned} \tag{3.8}$$

and

$$\psi(y) \equiv \phi'(y)/\phi(y). \tag{3.9}$$

We can eliminate $s^2(t)$ from the system (3.8) to get only two conditions

$$B(y) = 2A(y)\psi(y) + A'(y)/2 \tag{3.10}$$

and

$$C(y) = A(y)(\psi'(y) + \psi^2(y)) + \psi(y)A'(y)/2. \tag{3.11}$$

Clearly, by definition, $T_0 = 1$. Hence from (3.7) we get that $\deg(C) \leqslant 1$. Substituting $n = 1, 2$ into (3.7) we obtain analogously that $\deg(B) \leqslant 2, \deg(A) \leqslant 3$.

Theorem 1. *Assume that A, B, C are polynomials in y such that $\deg(A) \leqslant 3, \deg(B) \leqslant 2, \deg(C) \leqslant 1$ and that for at least one of these polynomials the strict equality holds (say, $\deg(A) = 3$). Define $T_0 = 1$ and construct a set of polynomials $T_n(y)$ through the recurrence relation (3.7). Then the resulting polynomials $T_n(y)$ have exact degree n and the Stieltjes ansatz (3.2) holds.*

In what follows we will assume that $A(y)$ is generic polynomial of the third degree:

$$A(y) = \kappa(y - e_1)(y - e_2)(y - e_3) \tag{3.12}$$

with some constant κ and distinct roots $e_i \neq e_j$ if $i \neq j$.

From condition (3.10) we see that

$$\psi(y) = \phi'/\phi = \frac{B(y) - A'(y)/2}{2A(y)} = q_2(y)/A(y), \tag{3.13}$$

where $q_2(y) = B(y)/2 - A'(y)/4$ is a polynomial of degree $\leqslant 2$. Thus we can write

$$\psi(y) = \alpha_1/(y - e_1) + \alpha_2/(y - e_2) + \alpha_3/(y - e_3) \tag{3.14}$$

with some constants α_i. Substituting expression (3.14) for $\psi(y)$ into (3.11) we see that conditions (3.10) and (3.11) are compatible if and only if at least one of the following four conditions holds:

(i) $\alpha_1 = \alpha_2 = \alpha_3 = 1/2$;
(ii) two of α_i are zero, say $\alpha_3 = \alpha_2 = 0$ and the rest is $\alpha_1 = 1/2$;
(iii) one of α_i is zero, say $\alpha_1 = 0$ and the rest are $\alpha_2 = \alpha_3 = 1/2$;
(iv) $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Case (iv) is degenerated: if all α_i are zero, then $\psi(y) = \phi'/\phi = 0$ and $\phi(y) \equiv$ const. But then from (3.3) it follows that $T_n(y) \equiv 0, n = 1, 2, \ldots$ which contradicts to our assumptions that $T_n(y)$ is a polynomial of exact degree n.

We thus can restrict ourselves with cases (i)–(iii).

Before detailing the investigation of these cases we first note that without loss of generality we can assume that

$$e_1 + e_2 + e_3 = 0. \tag{3.15}$$

Indeed, the variable y is defined up to an arbitrary affine transformation: substitution $y \to \alpha y + \beta$ does not change the main property (3.7). We thus can shift y by any constant. If all roots e_i are distinct (as assumed) then it is always possible to find such shift leading to condition (3.15).

Now, assuming that (3.15) holds we easily find from (3.8)

$$\mathrm{d}y/\mathrm{d}t = \sqrt{\kappa(y-e_1)(y-e_2)(y-e_3)}. \tag{3.16}$$

Again using the scaling freedom in the variable y, we can assume that $\kappa = 4$. We then find the following solution to (3.16):

$$y(t) = \wp(t-t_0; g_2, g_3), \tag{3.17}$$

where t_0 is an arbitrary parameter and $\wp(t; g_2, g_3)$ is the standard elliptic Weierstrass function with the invariants g_3, g_3. Recall [22] that the Weierstrass function $\wp(t; g_2, g_3)$ is a solution to the equation

$$(\mathrm{d}y/\mathrm{d}t)^2 = 4y^3 - g_2 y - g_3 = 4(y-e_1)(y-e_2)(y-e_3). \tag{3.18}$$

In what follows we will assume $t_0 = 0$ because the time variable t is defined up to an arbitrary shift $t \to t + \text{const}$.

Thus we derived that in the general situation (i.e. $A(y)$ is a polynomial of the third degree with simple roots), the function $y(t)$ (i.e. the argument of polynomials $T_n(y)$) coincides with the Weierstrass function $\wp(t; g_2, g_3)$. The corresponding moments $c_n(t)$ are thus elliptic functions of the argument t.

3.1. *Case (i)*

If $\alpha_1 = \alpha_2 = \alpha_3 = 1/2$ we can take

$$\phi(y) = 2\sqrt{(y-e_1)(y-e_2)(y-e_3)} = \wp'(t). \tag{3.19}$$

From (3.8) we get for the polynomials $B(y)$ and $C(y)$

$$B = 18y^2 + 6(e_1e_2 + e_1e_3 + e_2e_3), \qquad C = 12y. \tag{3.20}$$

It is seen that $\deg(B) = 2$ and $\deg(C) = 1$.

The polynomials $T_n(y) = \tau_n y^n + \sigma_n y^{n-1} + O(y^{n-2})$ have the leading terms τ_n and σ_n. From the explicit expressions (3.20) and from the recurrence relation (3.7) we find

$$\tau_n = 2^{2n}(n+1)!(3/2)_n, \qquad \sigma_n = 0. \tag{3.21}$$

3.2. *Case (ii)*

If $\alpha_2 = \alpha_3 = 0$ and $\alpha_1 = 1/2$, then we have from (3.13) (to within an unessential common factor)

$$\phi(y) = \sqrt{y-e_1} = \sqrt{\wp(t) - e_1}. \tag{3.22}$$

From (3.8) we get for the polynomials $B(y)$ and $C(y)$

$$B = 10y^2 + 4e_1 y + 2\left(3e_2e_3 - e_1^2\right), \qquad C = 2y + e_1. \tag{3.23}$$

It is seen that $\deg(B) = 2$ and $\deg(C) = 1$. For the two leading coefficients of the polynomials $T_n(y)$ we have

$$\tau_n = 2^{2n} n!(1/2)_n, \qquad \sigma_n = 2^{2n-1} n!(1/2)_n e_1, \tag{3.24}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the shifted factorial (Pochhammer symbol).

3.3. *Case (iii)*

If $\alpha_1 = 0$ and $\alpha_2 = \alpha_3 = 1/2$ we have

$$\phi(y) = \sqrt{(y-e_2)(y-e_3)}. \tag{3.25}$$

From (3.8) we get for polynomials $B(y)$ and $C(y)$

$$B = 14y^2 - 4e_1 y + 2\left(e_3e_2 - 3e_1^2\right), \qquad C = 6y - 3e_1. \tag{3.26}$$

It is seen that $\deg(B) = 2$ and $\deg(C) = 1$.

For the leading two coefficients of polynomials $T_n(y)$ we have

$$\tau_n = 2^{2n} n!(3/2)_n, \qquad \sigma_n = -2^{2n-1} n!(3/2)_n e_1/e_3. \tag{3.27}$$

4. Relation with the Lamé equation and the Lamé polynomials

The polynomials $T_n(y)$ we have obtained have a remarkable relation with the Lamé polynomials in the theory of the Lamé equation [11, 22].

Return to relation (3.6) for the polynomials $T_n(y)$ and introduce the operator M_n by the formula:

$$M_n = L^2 - \mu_n - \nu_n y = (s(y)\partial_y)^2 - \mu_n - \nu_n y, \tag{4.1}$$

where μ_n, ν_n are some parameters (not depending on y). Define the space of S_n functions $Q_n(y)\phi(y), n = 0, 1, \ldots$, where $Q_n(y)$ are arbitrary polynomials of fixed degree n. From (3.6) it is seen that for generic parameters μ_n, ν_n the operator M_n transforms the space S_n into the space S_{n+1}. By an appropriate choice of the parameter ν_n it is possible to achieve a more strong result:

Theorem 2. *Assume that $\phi(y)$ belongs to one of cases (i)–(iii). If*

$$\nu_n = \tau_{n+1}/\tau_n, \qquad \mu_n \neq \sigma_{n+1} - \frac{\tau_{n+1}\sigma_n}{\tau_n}, \tag{4.2}$$

where τ_n, σ_n are the two leading coefficients of the polynomials $T_n(y)$, then the operator M_n transforms the space S_n into itself.

Proof of this theorem is almost elementary and we omit it.

From this theorem it follows that for appropriate values of the parameter μ_n the operator M_n has kernel solutions

$$M_n\{W_n(y)\phi(y)\} = 0, \tag{4.3}$$

where $W_n(y)$ are some polynomials of degree n. Indeed, let us present $W_n(y)$ as $W_n(y) = \xi_{nk}T_k(y)$. Then relation (4.3) is equivalent to a system of linear equations:

$$\sum_{k=0}^{n} \xi_{nk}(T_{k+1}(y) - \nu_n y T_k(y)) = \mu_n \sum_{k=0}^{n} \xi_{nk}T_k(y). \tag{4.4}$$

But (4.4) is an eigenvalue problem in a linear space of dimension $n+1$ with basis vectors $T_0(y), T_1(y), \ldots, T_n(y)$. In this problem, $W_n(y)$ is an eigenvector and μ_n an eigenvalue. From linear algebra it is known that at least one eigenvector and corresponding eigenvalue always exists. Thus the functions $W_n(y)\phi(y)$ belong to the kernel of the operator M_n for appropriate values of μ_n.

In order to clarify the meaning of the polynomials $W_n(y)$ we return to the variable t. Then, obviously,

$$M_n = \partial_t^2 - \mu_n - \nu_n \wp(t) \tag{4.5}$$

and the eigenvalue problem for polynomials $W_n(y)$ is now rewritten in the form

$$\partial_t^2 W_n(\wp(t)) - \nu_n \wp(t) W_n(\wp(t)) = \mu_n W_n(\wp(t)). \tag{4.6}$$

But (4.7) coincides with the Lamé equation [11, 22]. Recall that in one of its forms, the Lamé equation can be written as

$$\frac{\mathrm{d}^2\Lambda(t)}{\mathrm{d}t^2} - (N(N+1)\wp(t) + B)\Lambda = 0 \tag{4.7}$$

for some unknown function $\Lambda(t)$. The parameter N is usually chosen to be integer whereas B is the 'eigenvalue' parameter.

In our case we have $N(N+1) = \nu_n = \tau_{n+1}/\tau_n$. Using explicit expressions for τ_n we have

Case (i). $N = 2n + 3$;
Case (ii). $N = 2n + 1$;
Case (iii). $N = 2n + 2$.

In the theory of the Lamé equation it is shown that only for such values of N there exist polynomial solutions $\Lambda(t) = W_n(\wp(t))\phi(t)$, where $\phi(t)$ has expressions (3.22), (3.25) and (3.19) [11]. Such polynomials $W_n(y)$ are called the *Lamé polynomials*. (There is also a fourth case of Lamé polynomials when $\phi(t) = 1$. This corresponds to our case (i) which was seen to be degenerate in our problem.)

We thus see that the polynomials $T_n(y)$ are intimately connected with the Lamé polynomials. Moreover, the operator L_n coincides (under appropriate choice of the parameter ν_n) with the Lamé operator. The Lamé polynomials $W_n(y)$ coincide with eigenvectors of the operator M_n. This allows us to propose a method to construct the Lamé polynomials starting from polynomials $T_n(y)$. Indeed, assume that polynomials $T_0(y) = 1, T_1(y), \ldots, T_n(y)$ are already explicitly constructed by recurrence relation (3.7). We can then obtain their structure coefficients η_{ks} in the expansion

$$yT_k(y) = \tau_k \nu_k^{-1} T_{k+1}(y) + \sum_{s=0}^{k} \eta_{ks} T_s(y), \qquad s = 0, 1, \ldots, n.$$

Then relation (4.4) is reduced to the linear system

$$\sum_{k=0}^{n} \xi_{nk}\zeta_{ks} = \mu_n \xi_{ns} \tag{4.8}$$

with some explicitly known coefficients ζ_{ks}. Finding of the coefficients ξ_{nk} and μ_n for the Lamé polynomials is thus equivalent to solving the ordinary eigenvalue problem (4.8), where $\xi_{nk}, k = 0, 1, \ldots, n$ are eigenvectors and μ_n eigenvalues.

5. Toda chain solutions and corresponding orthogonal polynomials. Case (i)

In this section we construct Toda chain solutions for the class (i), i.e. when

$$\phi(y) = 2\sqrt{(y-e_1)(y-e_2)(y-e_3)} = \wp'(t).$$

This means that $c_0(t) = \wp'(t)$. Thus for all moments we have obviously

$$c_n(t) = \wp^{(n+1)}(t). \tag{5.1}$$

There is a remarkable Kiepert formula for the corresponding Hankel determinants $D_n(t)$ [7, 22]:

$$D_n(t) = (-1)^n (1!2!\cdots n!)^2 \frac{\sigma(t(n+1))}{\sigma^{(n+1)^2}(t)}, \tag{5.2}$$

where $\sigma(t)$ is the standard sigma-function of Weierstrass [22] related to $\wp(t)$ by

$$\wp(z) = -\frac{d^2 \log \sigma(z)}{dz^2} \tag{5.3}$$

and

$$\wp(z) - \wp(y) = \frac{\sigma(z+y)\sigma(y-z)}{\sigma^2(z)\sigma^2(y)}. \tag{5.4}$$

From (5.4) it follows that

$$\wp'(z) = \frac{\sigma(2z)}{\sigma^4(z)}. \tag{5.5}$$

Recall also that [22]

$$\zeta(z) = \frac{d \log \sigma(z)}{dz}, \tag{5.6}$$

where $\zeta(z)$ is the Weierstrass zeta function.

It follows that the corresponding orthogonal polynomials $P_n(x;t)$ have the moments

$$c_n(t) = \wp^{(n+1)}(t), \qquad n = 0, 1, \ldots. \tag{5.7}$$

Hence the recurrence coefficients of these polynomials should satisfy Toda chain equations (1.1). Explicitly, these coefficients are easily calculated from (1.18) and (1.10). We have

$$h_n(t) = \frac{D_{n+1}(t)}{D_n(t)} = -(n+1)!^2 \frac{\sigma(t(n+2))}{\sigma(t(n+1))} \sigma(t)^{-2n-3} \tag{5.8}$$

whence

$$\begin{aligned} u_n(t) = h_n(t)/h_{n-1}(t) &= (n+1)^2 \frac{\sigma(t(n+2))\sigma(tn)}{\sigma^2(t(n+1))\sigma^2(t))} \\ &= (n+1)^2(\wp(t) - \wp(t(n+1))) \end{aligned} \tag{5.9}$$

and

$$b_n(t) = \frac{d \log(h_n(t))}{dt} = (n+2)\zeta(t(n+2)) - (n+1)\zeta(t(n+1)) - (2n+3)\zeta(t), \tag{5.10}$$

where we used (5.6).

Although the Toda chain relations (1.1) follow automatically from ansatz (5.7) it is instructive to verify them directly from the given coefficients $u_n(t), b_n(t)$. It is not difficult to do so using well-known formulae for the derivatives of the Weierstrass functions $\sigma(z), \zeta(z)$

Note that $u_0(t) = 0$ from (5.9), so we indeed deal with the restricted Toda chain solutions.

Solution (5.9) and (5.10) correspond to explicit elliptic continued fraction expansion found in [6]. Corresponding orthogonal polynomials $P_n(x;t)$ cannot have a positive orthogonality measure on the real axis. Indeed, in order for orthogonal polynomials $P_n(x;t)$ to have such a measure it is necessary and sufficient that the recurrence coefficients $b_n(t), u_n(t)$ be real and moreover, that the coefficient $u_n(t)$ be positive for all $n = 1, 2, \ldots$ [5]. Equivalently, this means that all Hankel determinants $D_n(t)$ should be positive $D_n(t) > 0$ for all $n = 0, 1, 2, \ldots$. But formula (5.2) indicates that this condition is impossible for any real value time t. Thus in this case the polynomials $P_n(x;t)$ can be orthogonal only on some contours in the complex domain.

6. Toda chain solutions and corresponding orthogonal polynomials: case (ii)

In this section we consider the Toda chain solutions corresponding to case (ii).

In order to calculate the recurrence coefficients $u_n(t), b_n(t)$ for orthogonal polynomials we need an explicit expression for the determinant

$$D_n = \det \left| c_0^{(i+k)}(t) \right|_{i,k=0\ldots,n-1}, \tag{6.1}$$

where

$$c_0(t) = \sqrt{\wp(t) - e_1} = \frac{\sigma_1(t)}{\sigma(t)} = \frac{\mathrm{e}^{-\eta_1 t}\sigma(t+\omega_1)}{\sigma(\omega_1)\sigma(t)}. \tag{6.2}$$

In (6.2) we exploited a well-known formula for presenting the Weierstrass function in terms of sigma functions [22]. It will be useful to recall the standard definition of the Weierstrass sigma functions [22]

$$\sigma_r(t) = \frac{\mathrm{e}^{-\eta_r t}\sigma(t+\omega_r)}{\sigma(\omega_r)}, \tag{6.3}$$

where $2\omega_1, 2\omega_2$ are fundamental periods of the Weierstrass function $\wp(t), \omega_3 = -\omega_1 - \omega_2$ and the parameters η_r are defined as

$$\eta_r \equiv \zeta(\omega_r), \qquad r = 1, 2, 3. \tag{6.4}$$

There is a fundamental relation between these parameters [22]

$$\eta_1\omega_2 - \eta_2\omega_1 = \mathrm{i}\pi/2. \tag{6.5}$$

We need also the following.

Lemma 1. *Assume that*

$$c_0(t) = \frac{\sigma(wt+q)}{\sigma(wt)\sigma(q)} \exp(\mu_1 t + \mu_0), \tag{6.6}$$

where w, q, μ_1, μ_0 are arbitrary parameters.

Then the Hankel determinant of the type (6.1) is

$$D_n(t) = \kappa_n \frac{\sigma(nwt+q)}{\sigma(wt)^{n^2}\sigma(q)} \exp(n(\mu_1 t + \mu_0)), \tag{6.7}$$

where

$$\kappa_n = 1!^2 2!^2 \cdots (n-1)!^2 w^{n(n-1)}. \tag{6.8}$$

Proof of this lemma can be obtained either by induction or directly from the determinantal formula found by Frobenius [2, 8] using a limiting process.

Using lemma 1 we can calculate the Hankel determinant (6.1) constructed from the moments (6.2):

$$D_n(t) = 1!^2 2!^2 \cdots (n-1)!^2\, \mathrm{e}^{-\eta_1 nt} \frac{\sigma(nt+\omega_1)}{\sigma(t)^{n^2}\sigma(\omega_1)} = 1!^2 2!^2 \cdots (n-1)!^2 \frac{\sigma_1(nt)}{\sigma(t)^{n^2}}. \tag{6.9}$$

Now we are able to calculate the normalization constants

$$h_n \equiv \frac{D_{n+1}(t)}{D_n(t)} = n!^2 \mathrm{e}^{-\eta_1 t} \frac{\sigma((n+1)t+\omega_1)}{\sigma(nt+\omega_1)\sigma^{2n+1}(t)} \tag{6.10}$$

and the recurrence coefficients

$$u_n(t) = h_n/h_{n-1} = n^2 \frac{s_{n+1}(t)s_{n-1}(t)}{s_n^2(t)\sigma^2(t)} = n^2(\wp(t) - \wp(nt+\omega_1)) \tag{6.11}$$

(where we put for brevity $s_n(t) \equiv \sigma(nt + \omega_1)$)

$$b_n(t) = \dot{h}_n/h_n = -\eta_1 + (n+1)\zeta((n+1)t + \omega_1) - n\zeta(nt + \omega_1) - (2n+1)\zeta(t). \tag{6.12}$$

Analogously one can obtain similar expressions if one replaces ω_1 by ω_2 and ω_3 (condition $\omega_1 + \omega_2 + \omega_3 = 0$ is assumed). Moreover, we can shift the variables t and y by arbitrary parameters. In order to classify possible choices of the function $c_0(t)$ we recall the formula [22]

$$\wp(t + \omega_\alpha) = e_\alpha + \frac{(e_\alpha - e_\beta)(e_\alpha - e_\gamma)}{\wp(t) - e_\alpha}, \tag{6.13}$$

where α, β, γ can take the values (1, 2, 3) or (2, 3, 1) or (3, 1, 2). Using this formula we can arrive at 12 possible expressions for the function $c_0(t)$ (up to a constant factor):

$$c_0(t) = \sqrt{\wp(t) - e_\alpha}, \qquad \frac{1}{\sqrt{\wp(t) - e_\alpha}}, \qquad \alpha = 1, 2, 3 \tag{6.14}$$

and

$$c_0(t) = \sqrt{\frac{\wp(t) - e_\alpha}{\wp(t) - e_\beta}}, \qquad \alpha, \beta = 1, 2, 3, \quad \alpha \neq \beta. \tag{6.15}$$

Equivalently, $c_0(t)$ can be presented in terms of the Jacobi elliptic functions as well, due to the well-known formulae [22]:

$$\sqrt{\wp(t) - e_3} = \frac{\nu}{\mathrm{sn}(\nu t; k)}, \qquad \sqrt{\wp(t) - e_2} = \frac{\nu\, \mathrm{dn}(\nu t; k)}{\mathrm{sn}(\nu t; k)}, \qquad \sqrt{\wp(t) - e_1} = \frac{\nu\, \mathrm{cn}(\nu t; k)}{\mathrm{sn}(\nu t; k)} \tag{6.16}$$

where

$$\nu = \sqrt{e_1 - e_3}, \qquad k^2 = \frac{e_2 - e_3}{e_1 - e_3}.$$

Combining these formulae with all 12 choices (6.14) and (6.15) we arrive at 12 possible Glaisher functions

$$c_0(t) = \mathrm{sn}(t), \mathrm{cn}(t), \mathrm{dn}(t), \mathrm{ns}(t), \mathrm{nc}(t), \mathrm{nd}(t) \tag{6.17}$$

and

$$c_0(t) = \mathrm{cs}(t), \mathrm{sc}(t), \mathrm{cd}(t), \mathrm{dc}(t), \mathrm{sd}(t), \mathrm{ds}(t). \tag{6.18}$$

Recall that Glaisher defined [22] his 12 types of the elliptic Jacobi functions in the following way: $\mathrm{ns}(t) = 1/\mathrm{sn}(t), \mathrm{nc}(t) = 1/\mathrm{cn}(t), \mathrm{nd}(t) = 1/\mathrm{dn}(t)$ and e.g. $\mathrm{cd}(t) = \mathrm{cn}(t)/\mathrm{dn}(t)$ (and the same rule for other possible combinations). In what follows we will assume that all parameters e_i are real and $e_1 > e_2 > e_3$. This condition guarantees that $0 < k^2 < 1$ and that the period $2\omega_1 = 2K(k)/\sqrt{e_1 - e_3}$ is real and the second period $2\omega_2 = 2\mathrm{i}K'(k)/\sqrt{e_1 - e_3}$ purely imaginary. Thus the fundamental parallelogram in his case is a rectangle [1].

Among the possible 12 choices for the function $c_0(t)$ there are two remarkable cases leading to orthogonal polynomials with a *positive* discrete measure on the real axis [25].

The first choice is

$$c_0(t) = \mathrm{nc}(wt) = \frac{1}{\mathrm{cn}(wt; k)}$$

with an arbitrary positive real parameter w.

In the admissible interval $-K/w < t < K/w$ we have the orthogonality relation

$$\sum_{s=-\infty}^{\infty} M_s(t) P_n(x_s; t) P_m(x_s; t) = h_n(t)\delta_{nm}, \tag{6.19}$$

where the spectral points x_s are located on the following uniform grid on the whole real axis:

$$x_n = \frac{\pi w}{2K'}(2n-1), \qquad n = 0, \pm 1, \pm 2, \ldots \tag{6.20}$$

and the corresponding masses are

$$M_n(t) = \frac{\pi}{k'K'} \frac{\exp(\pi w t(n-1/2)/K')}{v^{n-1/2} + v^{1/2-n}}, \qquad v = \exp(-\pi K/K'). \tag{6.21}$$

The second choice is

$$c_0(t) = \mathrm{dc}(wt) = \frac{\mathrm{dn}(wt; k)}{\mathrm{cn}(wt; k)}.$$

In this case the grid is again uniform

$$x_n = \frac{\pi w n}{K'}, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{6.22}$$

and the concentrated masses are

$$M_n(t) = \frac{2\pi}{K'(v^n + v^{-n})} \exp(\pi w n t/K'). \tag{6.23}$$

The admissible interval for t is the same as for the first case (6.19).

In both cases the concentrated masses are positive and the moment problem is determinate [25]. For a special choice of the time $t = 0$ we obtain the famous Stieltjes–Carlitz polynomials connected with elliptic functions [5]. For $t \neq 0$ we have a nontrivial generalization of these polynomials constructed in [25]. We see that these elliptic polynomials appear naturally as a special case of the ansatz (3.2).

7. Toda chain solutions and corresponding orthogonal polynomials: case (iii)

Consider the third possibility when

$$c_0(t) = \sqrt{(\wp(t) - e_2)(\wp(t) - e_3)}. \tag{7.1}$$

It is easily seen that the function defined by (7.1) is the derivative of the function $c_0(t) = \sqrt{\wp(t) - e_1}$ corresponding to case (ii) already considered. This means that the moments $c_n(t)$ corresponding to the choice (7.1) are obtained by the simple shift $c_n \to c_{n+1}$ from the moments corresponding to the choice (ii) (6.2). In turn, from the observations of section 2, this shift is seen to be equivalent to the simple Christoffel transform of the orthogonal polynomials $P_n(x; t)$ corresponding to the choice (ii).

Thus if we denote by $\tilde{P}_n(x; t)$ the polynomials corresponding to case (iii) then

$$\tilde{P}_n(x; t) = \frac{P_{n+1}(x; t) - A_n(t) P_n(x; t)}{x}, \tag{7.2}$$

where

$$A_n(t) = \frac{P_{n+1}(0; t)}{P_n(0; t)}.$$

For the corresponding Hankel determinants we have

$$\tilde{D}_n = (-1)^{n+1} D_n P_n(0; t).$$

Unfortunately, the explicit expression for the value $P_n(0; t)$ is unknown, so this formula does not give us an explicit (say, in terms of elliptic functions) formula for the Hankel determinant $\tilde{D}_n$. In contrast to cases (i) and (ii), the recurrence coefficients $\tilde{u}_n(t), \tilde{b}_n(t)$ perhaps do not have a nice explicit expression too.

8. Degenerate cases

So far, we have assumed that $A(y)$ in (3.7) is a polynomial of the third degree with distinct zeros e_1, e_2, e_3. This leads to elliptic solutions of the Stieltjes ansatz (3.2). In this section we consider briefly the degenerate case. This means that either some zeros e_i may coincide with one another, or $A(y)$ is a polynomial of degree less than 3.

Simple technical details will be omitted and we present final results only.

We first consider the case when the polynomial $A(y)$ is of the third degree with multiple zeros. If only two zeros coincide then we obtain solutions $c_0(t)$ and $y(t)$ which can be expressed in terms of hyperbolic or trigonometric functions. They correspond to the Krawtchouk, Meixner and Pollaczek polynomials (see [17] for details).

There are also additional solutions corresponding to the Christoffel transform (2.1) of these polynomials.

When all three zeros of the polynomial $A(y)$ coincide then we obtain solutions corresponding to the Laguerre polynomials.

Finally consider the case when the degree of the polynomial $A(y)$ is less than 3. Only the case $\deg(A(y)) = 1$ is compatible with the Stieltjes ansatz. This leads to solution (up to an affine transformation of the variable t) $c_0(t) = \exp(t^2/4)$, $y(t) = t^2$ corresponding to the Hermite polynomials. We have also an additional solution $c_0(t) = t\exp(t^2/4)$ corresponding to the Christoffel transform (2.1) of the Hermite polynomials.

All these solutions (apart from additional solutions corresponding to Christoffel transforms) correspond to the case when the Stieltjes ansatz can be reduced to ansatz (3.1). As was shown in [17] this ansatz corresponds to the orthogonal polynomials of the Sheffer class.

Let us consider a simple example connected with so-called derivative polynomials for tangent and secant [10]. It is well known that

$$\frac{d^n}{dt^n}\sec t = Q_n(\tan t)\sec t, \tag{8.1}$$

where $Q_n(z)$ are n-degree polynomials in z called the derivative polynomials for secant [10]. As was demonstrated in [10] these derivative polynomials appear in the expression for an improper integral

$$\int_{-\infty}^{\infty}\frac{x^n\,e^{ax}}{e^x+1}\,dx = \pi^{n+1}\csc a\pi\, Q_n(-\cot a\pi), \qquad n = 0, 1, 2, \ldots \tag{8.2}$$

where $0 < a < 1$.

We can interpret this result in terms of corresponding orthogonal polynomials with moments satisfying the Stieltjes ansatz. Indeed, formula (8.1) can be specialized for the even n as

$$\frac{d^{2n}}{dt^{2n}}\sec t = T_n(\tan^2 t)\sec t, \tag{8.3}$$

where $T_n(z^2) = Q_{2n}(z)$ (indeed, it is easy to verify that polynomials $Q_{2n}(z)$ contain only even degrees of z). Thus we have a special case of the Stieltjes ansatz (3.1) with $c_0(t) = \sec t$ and $y(t) = \tan^2 t$. From (3.8) we have

$$A(y) = \dot{y}^2 = 4y(1+y)^2, \qquad B(y) = 2(y+1)(5y+1), \qquad C(y) = 2y+1$$

which corresponds to a degenerate case (polynomial $A(y)$ has a double zero).

Take $c_0 = \sec t$ and $b_0 = \dot{c}_0/c_0 = \tan t, u_0 = 0$. Then we can apply the Toda chain equations (1.1) to obtain step-by-step $u_1 = \dot{b}_0 = \sec^2 t, b_1 = b_0 + \dot{u}_1/u_1 = 3\tan t, \ldots$. By induction, it is elementary verified that for any $n \geqslant 0$ we have

$$u_n = n^2\sec^2 t, \qquad b_n = (2n+1)\tan t. \tag{8.4}$$

Corresponding orthogonal polynomials $P_n(x;t)$ belong to a special case of the Meixner–Pollaczek polynomials. Indeed, the Meixner–Pollaczeck polynomials depend on two parameters λ, ϕ and have the recurrence coefficients [14]

$$u_n = \frac{n(n+2\lambda-1)}{4\sin^2\phi}, \qquad b_n = -\frac{n+\lambda}{\tan\phi}. \tag{8.5}$$

They are orthogonal on the whole real axis with respect to the weight function

$$w(x) = \mathrm{e}^{(2\phi-\pi)x}|\Gamma(\lambda+\mathrm{i}x)|^2. \tag{8.6}$$

If one puts $\lambda = 1/2$ and $\phi = \pi/2+t$ we obtain that recurrence coefficients (8.4) coincide with (8.5) (up to a scaling factor). The weight function (8.6) then becomes

$$w(x;t) = \frac{\mathrm{e}^{2tx}}{\cosh\pi x}$$

and formula (8.2) can be interpreted as explicit expression of the moments $c_n(t)$ in terms of the polynomials $Q_n(y)$.

This example can be generalized if one takes $c_0(t) = \sec^a t$ with an arbitrary positive parameter a. It is easily verified that the Stieltjes ansatz (3.2) holds with

$$A(y) = 4y(1+y)^2, \qquad B(y) = 2(1+y)(1+(2a+3)y), \qquad C(y) = a(1+(a+1)y).$$

Constructing corresponding Toda chain solutions we obtain by induction

$$u_n = n(n+a-1)\sec^2 t, \qquad b_n = (2n+a)\tan t \tag{8.7}$$

These recurrence coefficients correspond to generic Meixnner–Pollaczek polynomials with $\lambda = a/2, \phi = \pi/2+t$.

In fact, Stieltjes himself considered this example in details in his work [19].

If one chooses $c_0(t) = \sin t\sec^a t$ (this corresponds to the shift $c_0 \to c_1$ with an obvious shift of the parameter a) then we obtain Christoffel transformed Meixner–Pollaczek polynomials (2.1). In general their recurrence coefficients $u_n(t), b_n(t)$ will not have simple elementary expressions.

One can replace trigonometric functions with hyperbolic ones, say $c_0(t) = \cosh^N t, N = 1, 2, \ldots$ with $y(t) = \tanh^2 t$. This choice corresponds to the Krawtchouk polynomials. Indeed corresponding Toda chain solution is

$$u_n = \frac{n(n-N+1)}{\cosh^2 t}, \qquad b_n = (n-2N)\tanh t, \tag{8.8}$$

which coincides with recurrence coefficients for the Krawtchouk polynomials [14], whereas the additional solution $c_0(t) = \sinh t\cosh^N t$ corresponds to Christoffel transformed Krawtchouk polynomials (2.1).

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Note added in proof. As Suslov S has kindly pointed out to us; the derivative polynomials like in (8.1) are closely related with so-called Bateman and Pasternack orthogonal polynomials. See Koelink T H 1996 On Jacobi and continuous Hahn polynomials *Proc. Amer. Math. Soc.* **124** 887–898 for details concerning these polynomials. The authors thank Suslov S for bringing this reference to their attention.

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